Non-singular big-bounces and evolution of linear fluctuations

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We consider evolutions of *linear* fluctuations as the background Friedmann world model goes from contracting to expanding phases through smooth and non-singular bouncing phases. As long as the gravity dominates over the pressure gradient in the perturbation equation the growing-mode in the expanding phase is characterized by a conserved amplitude, we call it a *C*-mode. In the spherical geometry with a pressureless medium, we show that there exists a special gauge-invariant combination Φ which stays constant throughout the evolution from the big-bang to the big-crunch with the same value even after the bounce: it characterizes the coefficient of the *C*-mode. We show this result by using a bounce model where the pressure gradient term is negligible during the bounce; this requires additional presence of an exotic matter. In such a bounce, even in more general situations of the equation of states before and after the bounce, the *C*-mode in the expanding phase is affected only by the *C*-mode in the contracting phase, thus the growing mode in the contracting phase decays away as the world model enters expanding phase. In the case the background curvature has significant role during the bounce, the pressure gradient term becomes important and we cannot trace *C*-mode in the expanding phase to the one before the bounce. In such situations, perturbations in a fluid bounce model show exponential instability, whereas the ones in a scalar field bounce model show oscillatory behaviors.

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I. INTRODUCTION

The collapsing and bouncing phases of the FLRW (Friedmann-Lemaître-Robertson-Walker) world models we are considering are the possible ones in our past, before the big-bang. The same physics, however, could work in the possible case in future as well.

The re-expansion of a positive curvature Friedmann world model which is destined to collapse, or cyclic repetition of the process with such a bounce was proposed as early as in the 1930's [1–3]. Specific realizations of the bounce and the conditions required to have the FLRW world model from a bounce were studied in [4,5]. The singularity-free cosmologies are possible as we give up the strong energy condition which is often possible with quantum corrections [4]. Recently, the big-bang world model preceded by a collapsing phase attracted renewed attention in the context of the brane cosmology [6].

In this paper we analyse the evolution of scalar-type curvature (often called the adiabatic) fluctuations as the background world model goes through a smooth and non-singular bounce which connects the contracting and the expanding phases. We will *assume* the classical General Relativity is valid as the correct gravity theory throughout the evolution, and also consider scales where the linear approximation is valid.

In §II we review the cosmological perturbation theory needed for our analyses in later sections. In §III we present the large-scale evolutions of various curvature perturbations near singularity. In §IV we analyse the evolution using exact solutions in a pressureless situation. In §V we show the evolution of perturbation through a bounce model using three different bounce models. §VI presents a summary, implications of our work, and discussions on related works. We set $c \equiv 1$.

II. COSMOLOGICAL PERTURBATIONS

Our metric convention is [7,8]

$$ds^2 = -a^2(1 + 2\alpha)d\eta^2 - 2a^2(\beta_{,\alpha} + B_\alpha^{(v)})d\eta dx^\alpha$$
$$+ a^2\Big[g_{\alpha\beta}^{(3)}(1 + 2\varphi) + 2\gamma_{,\alpha|\beta} + 2C_{(\alpha|\beta)}^{(v)}$$
$$+ 2C_{\alpha\beta}^{(t)}\Big]dx^\alpha dx^\beta. \tag{1}$$

The perturbed order variables α, β, φ and γ are scalar-type perturbations; the transverse $B_\alpha^{(v)}$ and $C_\alpha^{(v)}$ are vector-type perturbations (rotation); a transverse-tracefree $C_{\alpha\beta}^{(t)}$ is tensor-type perturbation (gravitational wave). The energy-momentum tensor is

$$T_0^0 \equiv -(\bar{\mu} + \delta\mu),$$
$$T_\alpha^0 \equiv (\mu + p)\Big[-(1/k)v_{,\alpha} + v_\alpha^{(v)}\Big],$$
$$T_\beta^\alpha \equiv (\bar{p} + \delta p)\delta_\beta^\alpha + \pi_\beta^\alpha, \tag{2}$$

where the tracefree π_β^α is the anisotropic stress.

The trace and tracefree parts of extrinsic curvature (equivalently, the expansion $\hat{\theta}$ and the shear $\hat{\sigma}_{ab}$ of the

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normal frame vector field), and the intrinsic scalar curvature $R^{(h)}$ of the constant-time spacelike hypersurface are, see eqs. (C3,C14) of [8] and eqs. (A6,A7) of [9],

$$\hat{\theta} = 3H - \kappa,$$

$$\hat{\sigma}_{\alpha\beta} = \chi_{,\alpha|\beta} - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\chi + a\Psi^{(v)}_{(\alpha|\beta)} + a^2\dot{C}^{(t)}_{\alpha\beta},$$

$$R^{(h)} = \frac{1}{a^2}\left[6K - 4(\Delta + 3K)\varphi\right], \tag{3}$$

where*

$$\chi \equiv a(\beta + a\dot{\gamma}), \quad \kappa \equiv 3(H\alpha - \dot{\varphi}) - \frac{\Delta}{a^2}\chi,$$

$$\Psi^{(v)}_\alpha \equiv B^{(v)}_\alpha + a\dot{C}^{(v)}_\alpha. \tag{4}$$

We have $H \equiv \frac{\dot{a}}{a}$, K the normalized background three-space curvature, and an overdot indicates time derivative based on t, $dt \equiv ad\eta$. Thus, κ, χ, and φ are the perturbed expansion, the scalar-type shear, and the perturbed three-space scalar-curvature of the normal hypersurface, respectively. $\Psi^{(v)}_\alpha$ and $\dot{C}^{(t)}_{\alpha\beta}$ give the vector- and tensor-type contributions to the shear tensor.

$C^{(t)}_{\alpha\beta}$, $\Psi^{(v)}_\alpha$ and $v^{(v)}_\alpha$ are gauge-invariant. α, φ, χ, κ, v, $\delta\mu$ and δp are spatially gauge-invariant but depend on the temporal gauge condition, i.e, depend on the spatial hypersurface (time slicing) choice [9]. Setting any one of these temporally gauge dependent variables equal to zero corresponds to a fundamental gauge condition; except for the synchronous gauge ($\alpha \equiv 0$) each of the other conditions fixes the temporal gauge degree of freedom completely, and any variable in such a gauge condition uniquely corresponds to a gauge-invariant combination (of the variable and the variable used in the gauge condition) [7,8].

Equations describing the evolution of a spatially homogeneous and isotropic FLRW world model are:

$$H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad \dot{\mu} = -3H(\mu + p). \tag{5}$$

The Λ term can be also considered as an ideal fluid with $\mu_\Lambda = -p_\Lambda = \Lambda/(8\pi G)$; in such cases, we have $\mu = \mu_m + \mu_\Lambda$ and $p = p_m + p_\Lambda$. In the case of a minimally coupled scalar field (ϕ) we have $\mu = \frac{1}{2}\dot{\phi}^2 + V$ and $p = \frac{1}{2}\dot{\phi}^2 - V$, with an equation of motion $\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$. Λ can be included as $V_\Lambda = \Lambda/(8\pi G)$.

We consider the general scalar-type perturbation. It is convenient to introduce [10–12]

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\mu + p)}\varphi_\chi. \tag{6}$$

The scalar-type perturbation of a fluid with vanishing anisotropic stress in the Einstein's gravity is described by [12]:

$$\Phi = \frac{H^2}{4\pi G(\mu + p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \tag{7}$$

$$\dot{\Phi} = -\frac{Hc_s^2}{4\pi G(\mu + p)}\frac{k^2}{a^2}\varphi_\chi - \frac{H}{\mu + p}e, \tag{8}$$

$$\frac{k^2 - 3K}{a^2}\varphi_\chi = 4\pi G\mu\delta_v, \tag{9}$$

where $w \equiv p/\mu$ and $c_s^2 \equiv \dot{p}/\dot{\mu}$. $\varphi_v \equiv \varphi - (aH/k)v$, $\varphi_\chi \equiv \varphi - H\chi$ and $\delta_v \equiv \delta + 3(aH/k)(1 + w)v$ are gauge-invariant combinations [7,8]; φ_v is the same as φ in the comoving gauge ($v \equiv 0$), and φ_χ is the same as φ in the zero-shear gauge ($\chi \equiv 0$), etc.

We emphasize that results in this section are valid considering general K, Λ, and time-varying equation of state $p(\mu)$. In the case of a minimally coupled scalar field, we have additional nonvanishing entropic perturbation (the isotropic stress), $e = (1 - c_s^2)\delta\mu_v$. Its effect can be covered by changing c_s^2 in eq. (8) to $c_A^2 \equiv 1 - 3(1 - c_s^2)K/k^2$, [13]; as eqs. (7-9) are a complete set for single component, this prescription applies always in the single scalar field case. It is convenient to have

$$\varphi_\delta \equiv \varphi + \frac{\delta}{3(1 + w)} = \Phi + \frac{k^2}{a^2}\frac{1}{12\pi G(\mu + p)}\varphi_\chi. \tag{10}$$

We can show in general [13]

$$\varphi_\kappa = \varphi_\delta \Big/ \left[1 + \frac{k^2 - 3K}{12\pi G(\mu + p)a^2}\right], \tag{11}$$

where $\varphi_\kappa \equiv \varphi + H\kappa/(3\dot{H} - k^2/a^2)$. In the notation of Bardeen [9] we have

$$\delta_v = \epsilon_m, \quad \varphi_\chi = \Phi_H, \quad \varphi_v = \phi_m, \quad \varphi_\kappa = \phi_h. \tag{12}$$

We have $\varphi_\delta = \zeta$ in [7], and $\varphi_v = \mathcal{R}$ in [14]; φ_v was also originally introduced by Lukash as $-\frac{1}{3}q$ in [15]. From eqs. (7,8) we can derive equations in closed forms:

$$\ddot{\bar{\Phi}} + \left(c_s^2k^2/a^2 - \ddot{x}/x\right)\bar{\Phi} = 0, \tag{13}$$

$$\ddot{\bar{\varphi}}_\chi + \left(c_s^2k^2/a^2 - \ddot{y}/y\right)\bar{\varphi}_\chi = 0, \tag{14}$$

where

$$\bar{\Phi} \equiv x\Phi, \quad \bar{\varphi}_\chi \equiv (ay/H)\varphi_\chi,$$

$$y \equiv H/\sqrt{(\mu + p)a} \equiv (a/c_s)x^{-1}. \tag{15}$$

Equations using the conformal time were presented in [16].

In the large-scale limit (meaning $c_s^2k^2/a^2$ term negligible, thus gravity dominates over pressure) we have general solutions [11–13]:

$$\Phi(k,t) = C(k) - d(k)\frac{k^2}{4\pi G}\int^t dt/x^2, \qquad (16)$$

$$\varphi_\chi(k,t) = 4\pi G C(k)\frac{H}{a}\int^t dt/y^2 + d(k)\frac{H}{a}, \qquad (17)$$

where C and d are the two spatially dependent integration constants: we call these the C-mode and the d-mode, respectively. Solutions for δ_v, φ_δ and φ_κ follow from eqs. (9-11). Notice that the d-mode of Φ is higher-order in the large-scale expansion compared with the d-mode of φ_χ. In a pressureless medium, the above solutions are exact, and we have $\Phi = C$ [10]. In fact, for such a medium, instead of eq. (13), eq. (8) gives $\dot{\Phi} = 0$.

In order to use the large-scale solutions in eqs. (16,17) it is important to check whether we could ignore the $c_s^2 k^2/a^2$ term during the evolution. The large-scale condition implies (pressure/gravity) $\ll 1$ where

$$\frac{\text{pressure}}{\text{gravity}} \quad \sim \quad \frac{c_s^2 k^2/a^2}{\ddot{x}/x}, \quad \frac{c_s^2 k^2/a^2}{\ddot{y}/y}. \qquad (18)$$

In a positive curvature (spherical) model the wave number varies as $k = \sqrt{(n^2-1)K}$ where $n = 1,2,3,\ldots$; $n = 1,2$ are known to be unphysical [17,9]. In a negative curvature (hyperbolic) model, $k > \sqrt{|K|}$ and $0 \le k < \sqrt{|K|}$ correspond to the subcurvature and the supercurvature scales, respectively [18]. In a zero curvature (flat) model we have $k \ge 0$.

The following two variables are continuous under a sudden jump of equation of state [19]:

$$\varphi_\chi \quad (\text{or } \delta_v), \quad \varphi_\delta. \qquad (19)$$

These joining variables work for general K, Λ, and $p(\mu)$ in the general scale. This applies for the perfect fluids, and for the cases involving scalar fields, see [19]. For the background, a and $\dot{a}$ should be continuous at the transition. Consider two phases I and II with different equation of states, w_I and w_{II}, making a transition at t_1. Assuming a flat background, in the large-scale limit, by matching φ_χ and φ_δ in eqs. (16,17,10) we can see that to the leading order in the large-scale expansion we have

$$C_{II} = C_I,$$
$$d_{II} = d_I + 4\pi G C_I\left[\left.\int^{t_1}\frac{a(\mu+p)}{H^2}dt\right|_I\right.$$
$$\left.- \int^{t_1}\frac{a(\mu+p)}{H^2}dt\right|_{II}\right]. \qquad (20)$$

Thus, to the leading order in the large-scale expansion the C-mode of Φ remains the same, whereas the d-mode of φ_χ is affected by the transition and also the previous history of the d- and C-modes [19]. Applications were made in [20].

Ignoring the anisotropic stress (π_β^α) and assuming $K = 0$, equations for the rotation and the gravitational wave become [8]

$$\left[a^4(\mu+p)v_\alpha^{(v)}\right]^{\cdot} = 0; \quad \frac{k^2}{a^2}\Psi_\alpha^{(v)} = 16\pi G(\mu+p)v_\alpha^{(v)}, \quad (21)$$

$$\ddot{\bar{C}}_{\alpha\beta}^{(t)} + \left(k^2/a^2 - \ddot{z}/z\right)\bar{C}_{\alpha\beta}^{(t)} = 0, \qquad (22)$$

where $\bar{C}_{\alpha\beta}^{(t)} \equiv z C_{\alpha\beta}^{(t)}$ and $z \equiv a^{3/2}$. The equation for $C_{\alpha\beta}^{(t)}$ satisfies the same equation as Φ in eq. (13) with $x \propto a^{3/2}$. Thus, in general scale for $\Psi_\alpha^{(v)}$ and in the large-scale limit for $C_{\alpha\beta}^{(t)}$ we have general solutions

$$\Psi_\alpha^{(v)}(k,t) = d_\alpha(k)\frac{1}{a^2}, \qquad (23)$$

$$C_{\alpha\beta}^{(t)}(k,t) = c_{\alpha\beta}(k) - d_{\alpha\beta}(k)\int^t dt/a^3. \qquad (24)$$

Similarly as the C-mode of Φ the amplitude of $c_{\alpha\beta}$-mode simply stays constant.

III. LARGE-SCALE EVOLUTIONS OF CURVATURE PERTURBATIONS

The general large-scale solutions for scalar- and tensor-type perturbations, and the general solutions for the vector-type perturbation are presented in eqs. (16,17,23,24). We call the solutions with C and $c_{\alpha\beta}$ the C-modes, and the solutions with d, d_α and $d_{\alpha\beta}$ the d-modes. The vector-type perturbation has no C-mode. In the expanding phase C-modes are relatively growing solutions whereas d-modes decay, thus transient, in time. In a contracting phase, however, the opposite is the case with the d-modes often diverging as the background model approaches the singularity.

In this section we assume near *flat* background. With a constant w we have $c_s^2 = w$, and $a \propto |t|^{\frac{2}{3(1+w)}}$ for $-1 < w \le 1$. In a medium with $w > -\frac{1}{3}$, as we approach the singularity $\frac{k}{aH}(\propto |t|^{\frac{1+3w}{3+3w}})$ becomes negligible for any given scale with k, thus the large-scale conditions are well satisfied. In such a case, during dynamical time-scale of the background evolution "light can travel only a small fraction of a wavelength" [9], thus the scale becomes super-horizon scale; Bardeen [9] called it ($\frac{k}{aH} \sim 1$) an "effective particle horizon"[†]. In general, we consider

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[†] The effective particle horizon is the same as the "Hubble sphere" studied in [21], and closely resembles the "z-surface" introduced in [22]. The global concepts like particle- and event-horizons are not suitable to describe the local dynamically reachable ranges. Studies in [21,22] show that the Hubble sphere and the z-surface are more suitable to describe the concepts like 'scales becomming super-horizon size during the inflation era'. Similarly, these are suitable to describe the same physics during contracting phase with $w > -\frac{1}{3}$. Under this situation we can show that to an observer in the contract-

$w \geq 0$: "a single-component treatment of the matter is inappropriate when the net pressure is negative" [9].

In this case, from eqs. (16,17,24) we have the C-modes remain constant in time

$$\varphi_\chi = \frac{3+3w}{5+3w}C, \quad \varphi_v = C, \quad C_{\alpha\beta}^{(t)} = c_{\alpha\beta}. \qquad (25)$$

Thus, in an expanding medium the perturbation evolutions in the super-effective-particle-horizon are kinematic[‡] and are characterized by the conserved quantities, see [24].

Meanwhile, the d-modes behave as

$$\varphi_\chi \propto |t|^{-\frac{5+3w}{3+3w}}, \quad \varphi_v, \; C_{\alpha\beta}^{(t)} \propto |t|^{-\frac{1-w}{1+w}},$$
$$\Psi_\alpha^{(v)} \propto |t|^{-\frac{4}{3(1+w)}}, \qquad (26)$$

for $-1 < w < 1$. For $w = 1$, the above solution is valid for $\varphi_\chi (\propto |t|^{-4/3})$ and $\Psi_\alpha^{(v)} (\propto |t|^{-2/3})$, whereas we have $\varphi_v, \; C_{\alpha\beta}^{(t)} \propto \ln |t|$ instead.

In the case of constant w, a complete set of solutions of the scalar-type perturbation in six different fundamental gauge conditions is presented in Tables 2-5 of [25]. Although the solutions in [25] are presented in the context of the expanding phase, by changing the time variables to their absolute values with the singularity at $|t| = 0$, the same solutions apply in the contracting phase as well.

A. Intrinsic curvature

φ is a dimensionless measure of the (intrinsic) curvature perturbation of the hypersurface (temporal gauge condition) we choose. Thus, its value depends on the chosen hypersurface (temporal gauge condition). From eq. (A6) of [9] we notice that $C_{\alpha\beta}^{(t)}$ gives a dimensionless contribution to the tensor-type intrinsic curvature perturbation, and the vector-type perturbation does not contribute to the curvature perturbation, see also eq. (C14) in [8]. Table 2 of [25] shows that for the d-mode[§]

$$\varphi_\kappa, \; \varphi_\alpha, \; \varphi_\delta, \; \varphi_v, \; C_{\alpha\beta}^{(t)} \propto |t|^{-\frac{1-w}{1+w}}, \quad \varphi_\chi \propto |t|^{-\frac{5+3w}{3+3w}}, \quad (27)$$

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ing phase, an object separated by a given comoving distance appears more blue-shifted as time goes on, i.e, the object is effectively receding from the observer.

[‡] This kinematic nature of the evolution is reflected in the conserved behaviors of linear perturbations in the expanding phase [9,7]. What is conserved is the the amplitude of the C-mode of curvature perturbation in diverse gauge conditions. In the expanding phase the situation is also well represented by the separated Friedmann picture of the perturbed Friedmann world model pioneered (up to our knowledge) by Harrison in [23].

[§] Although φ_α, the φ in the synchronous gauge ($\alpha \equiv 0$), is not gauge-invariant, we are considering the physical solutions.

for $-1 < w < 1$. For $w = 1$ we have[**], from Table 4 of [25],

$$\varphi_\kappa, \; \varphi_\alpha, \; \varphi_\delta, \; \varphi_v, \; C_{\alpha\beta}^{(t)} \propto \ln |t|, \quad \varphi_\chi \propto |t|^{-4/3}. \qquad (28)$$

Thus, even for $w = 1$, φ diverges in all gauge conditions considered. φ in the zero-shear gauge diverges more strongly compared with the ones in the other gauge conditions. The strong divergence in the zero-shear gauge is known to be due to the strong curvature of the hypersurface (temporal gauge condition) [9]. φ is set to zero in the uniform-curvature gauge. For the C-modes we have eq. (25) and $\varphi_\kappa = \varphi_\alpha = \varphi_\delta = \varphi_v = C$.

B. Extrinsic curvature

Although we have no shear in the background model, the perturbed scalar-type shear is still a gauge dependent quantity. A dimensionless measure of the shear variable (the shear divided by the background expansion rate) becomes

$$\frac{\hat{\sigma}}{H} \sim \left(\frac{k^2}{a^2 H^2} H\chi, \; \frac{k}{aH} \Psi_\alpha^{(v)}, \; \frac{1}{H} \dot{C}_{\alpha\beta}^{(t)} \right), \qquad (29)$$

for three perturbation types; $\hat{\sigma} \equiv \sqrt{\hat{\sigma}_{\alpha\beta} \hat{\sigma}^{\alpha\beta}/2}$. The d-modes of all-types of perturbations show *the same* temporal behavior

$$\frac{\hat{\sigma}}{H} \propto |t|^{-\frac{1-w}{1+w}}, \qquad (30)$$

for $-1 < w \leq 1$, thus, with no logarithmic divergences for $w = 1$ case. The result for vector- and tensor-type perturbations follow from eqs. (23,24), and the one for scalar-type perturbation follows from the Tables 2 and 4 of [25]; thus, the solutions apply to the gauge conditions considered (we set $\chi = 0$ in the zero-shear gauge). We can check that the C-modes contributions to $\hat{\sigma}/H$ are all regular near the singularity for $-1 < w \leq 1$. The behavior of κ/H, a dimensionless measure of the perturbed trace part of the extrinsic curvature, varies widely depending on the gauge conditions, see Tables 2 and 4 of [25].

C. Weyl curvature

Durrer has informed us another useful measure of the spacetime fluctuation which behaves regularly for $w = 1$, the Weyl curvature C_{abcd}. The Weyl-curvature (the conformal tensor) vanishes in the FLRW background geometry, and is naturally gauge-invariant. The Weyl tensor

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[**] One other special case occurs for $w = 0$ where we have no d-modes for φ_v and φ_α, see Table 5 of [25].

can be covariantly decomposed to the electric E_{ab} and magnetic H_{ab} parts, [26]. Using eq. (C9) in [8], see also eqs. (2.26,2.27) of [27], we can show

$$\frac{E}{R} \sim \left(\frac{k^2}{a^2 H^2} \varphi_\chi, \; \frac{k}{aH^2} \dot{\Psi}_\alpha^{(v)}, \; \frac{1}{H} \dot{C}_{\alpha\beta}^{(t)} \right), \tag{31}$$

where $E \equiv \sqrt{E^{\alpha\beta} E_{\alpha\beta}/2}$ and R is the scalar-curvature ($\sim H^2$). Thus, the d-modes of all-types of perturbations behave exactly like $\hat{\sigma}/H (\propto |t|^{-\frac{1-w}{1+w}})$, thus behave regularly for $w = 1$. H_{ab} only contributes to the vector- and tensor-type perturbations, and we can also show that the d-modes behave as $\sqrt{H^{\alpha\beta} H_{\alpha\beta}/2}/R \propto |t|^{-\frac{2}{3}\frac{1-3w}{1+w}}$, thus behave more regularly for $w = 1$. This regular behavior of the Weyl curvature at singularity for $w = 1$ case was used to argue the validity of perturbation theory in such a situation, see around eq. (5.20) of [28]; however, see our discussion below eq. (68).

IV. EXACT EVOLUTION IN A PRESSURELESS CASE

For $K > 0$, $\Lambda = 0$ and $p = 0$, eq. (5) gives a cycloid [29,2]

$$a = c_m (1 - \cos \eta), \quad t = c_m (\eta - \sin \eta), \tag{32}$$

where $c_m \equiv (4\pi G/3)\mu a^3$ and $d\eta \equiv dt/a$. K is normalized to unity, thus $0 \leq \eta \leq 2\pi$.

For $p = 0$, eqs. (9,14) give

$$\begin{aligned}
&\left[a^2 H^2 \left(\delta_v/H \right)^{\cdot} \right]^{\cdot} / (a^2 H) \\
&= \ddot{\delta}_v + 2H\dot{\delta}_v + 4\pi G\mu \delta_v = 0,
\end{aligned} \tag{33}$$

which coincides with the density perturbation equation in the synchronous gauge [17], or in the Newtonian context [30]; δ_v is the energy density perturbation in the comoving gauge, [31]. Assuming $\Lambda = 0$, the two independent *exact* solutions for $\varphi_\chi \propto \delta_v/a$ in eq. (17) are [32]:

$$\begin{aligned}
c_m \frac{H}{a} \int^t \frac{dt}{a^2 H^2} &= -\frac{3\eta \sin \eta}{(1 - \cos \eta)^3} + \frac{5 + \cos \eta}{(1 - \cos \eta)^2} \\
&\equiv \varphi_+(\eta)/3, \\
c_m^2 \frac{H}{a} &= \frac{\sin \eta}{(1 - \cos \eta)^3} \equiv \varphi_d(\eta).
\end{aligned} \tag{34}$$

In asymptotes we have

$$\begin{aligned}
\varphi_+(\eta) &\simeq 3/5, \quad \varphi_d(\eta) \simeq 8/\eta^5, \quad (\eta \ll 1), \\
\varphi_+(\eta)/(18\pi) &\simeq (8/\bar{\eta}^5) \simeq -\varphi_d(\eta), \quad (\bar{\eta} \ll 1),
\end{aligned} \tag{35}$$

where $\bar{\eta} \equiv 2\pi - \eta$. We have

$$\varphi_+(2\pi - \eta) = \varphi_+(\eta) + 18\pi\varphi_d(\eta) \equiv \varphi_-(\eta), \tag{36}$$

where φ_- shows time inverted evolution of φ_+, [33].

Equations (16,17,9-11) give exact solutions:

$$\Phi = C, \tag{37}$$

$$\varphi_\chi = \varphi_+ C + \varphi_d \tilde{d} = \frac{3}{k^2 - 3} \frac{\delta_v}{1 - \cos \eta}, \tag{38}$$

$$\varphi_\delta = C + (k^2/9)(1 - \cos \eta)\varphi_\chi, \tag{39}$$

and similarly for φ_κ; $\tilde{d} \equiv d/c_m^2$ is dimensionless. For $\eta \ll 1$ we have $\varphi_\chi = \frac{3}{5}C$ and $\varphi_\delta = \varphi_\kappa = C$ for the C-modes. From eq. (8) we have $\dot{\Phi} = 0$ exactly for a pressureless fluid considering general K and Λ. Φ has only the C-mode (it is identified as C), and no d-mode. Evolutions of φ_χ and Φ are shown in Fig. 1. Notice that both φ_+ and φ_d diverge as the model approaches the big crunch singularity.
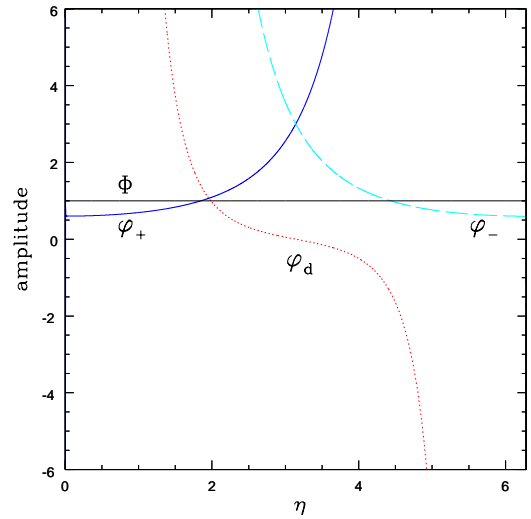

FIG. 1. Right: Evolutions of φ_+ (blue, line), φ_- (cyan, dashed line), φ_d (blue, dotted line), and Φ (black, horizontal line).

Let us consider a scenario where the big crunch is succeeded to an expanding phase: thus we have two phases $\eta \leq \eta_1$ (phase I) and $\eta \geq \eta_1$ (phase II) with $\eta_1 = 0$. For φ_χ we can take two out of three forms of solutions $(\varphi_+, \varphi_-, \varphi_d)$ as the general solutions in either phase. Although $\dot{a}$ could be discontinuous at the transition reaching the singularity let us try matching φ_χ and φ_δ directly at η_1; afterall we assume a nonsingular bounce near η_1, see later. Using eqs. (38,39,35) we can show

$$C_{II} = C_I, \quad \tilde{d}_{II} = \tilde{d}_I - 18\pi C_I. \tag{40}$$

This implies that the value of Φ variable is conserved even through the bounce. In terms of the general solutions we notice the following. Using eq. (36) we can decompose φ_+ in eq. (38) to φ_- and φ_d. Near bounce, although $\varphi_-^I \simeq \frac{3}{5}$ is negligible compared with $\varphi_+^I \simeq 18\pi(8/|\eta|^5)$ and $\varphi_d^I \simeq -8/|\eta|^5$, we can show that it is this constant

mode of φ_-^I which feeds the C-mode after the bounce. Thus, in the collapsing phase it is appropriate to write eq. (38) as

$$\varphi_\chi = \varphi_- C + \varphi_d(\tilde{d} - 18\pi C). \tag{41}$$

Therefore, if such a bounce is allowed, we have shown that φ_-^I feeds the growing mode φ_+^{II} in the expanding phase. The apparent growing (diverging) solutions φ_+^I or φ_d^I only feed the φ_d^{II} or φ_-^{II} which are the decaying mode in expanding phase. The time-scale of a cycle is encoded in c_m of eq. (32) which can affect only the decaying solution in the expanding phase. The value of Φ, which is C, is not affected by the different duration of each cycle.

Notice, however, that if we strictly consider the singular and cuspy bounce at η_1 implied by eq. (32) we have $\dot{a}$ discontinuous, which forbids us from relying on the matching conditions. We have assumed such a singular bounce can be regarded as a limiting case of a smooth and non-singular bounce; a concrete example will be considered in the next section. We note that the curvature term has negligible role near the big crunch/bang. We also have assumed the linearity of the fluctuations involved.

V. THROUGH THE BOUNCE

Assume two expanding phases I and II with equation of states w_I and w_{II}. In near flat situation we have

$$a(t) = a_{0i}(t - t_i)^{\frac{2}{3(1+w_i)}}, \tag{42}$$

where $i = I, II$. The coefficients should be determined by matching a and $\dot{a}$ at the transitions t_1. In phases I and II the large-scale solutions in eqs. (16,17,10) give for $-1 < w_i < 1$:

$$\Phi = C + k^2 \frac{4}{9} \frac{w_i}{1 - w_i^2} \frac{1}{a^3 H} d, \tag{43}$$

$$\varphi_\chi = \frac{3 + 3w_i}{5 + 3w_i} C + \frac{H}{a} d, \tag{44}$$

$$\varphi_\delta = C + k^2 \frac{2}{9} \frac{1}{1 - w_i} \frac{1}{a^3 H} d, \tag{45}$$

where C and d in the phase I should be regarded as C_I and d_I, and similarly for the phase II. For $w_i = 1$, d-mode of Φ (and part of φ_δ) contains $\ln(t - t_i)$ term instead of $(1 - w_i)^{-1}$. Equations (6,11) give φ_v and φ_κ. As long as we have a and $\dot{a}$ continuous through a transition from phase I to phase II at t_1 we can use our joining variables in eq. (19). Examples are the radiation-matter transition and the inflation-radiation transition. Using eqs. (44,45), to the leading order in the large-scale expansion we have

$$C_{II} = C_I,$$
$$d_{II} = d_I + \frac{6(w_I - w_{II})}{(5 + 3w_I)(5 + 3w_{II})} \frac{a_1}{H_1} C_I. \tag{46}$$

This is consistent with the result derived in eq. (17) of [19]. Similar results hold for two contracting phases as well.

In the case of transition from the contracting to expanding phases, however, $\dot{a}$ can be discontinuous at the transition. In order to handle the case properly, we need an intermediate bouncing phase B which smoothly connects the two phases I and II. We consider the collapsing (I) and expanding (II) phases smoothly connected by a nonsingular bouncing phase (B). Assuming the curvature is not important in phases I and II near the bounce, and assuming w_I and w_{II} for the two phases we have

$$a_I(t) = a_{I0}[-(t - t_I)]^{\frac{2}{3(1+w_I)}}, \tag{47}$$

$$a_{II}(t) = a_{II0}(t - t_{II})^{\frac{2}{3(1+w_{II})}}. \tag{48}$$

The coefficients should be determined by matching a and $\dot{a}$ at the transitions t_1 and t_2.

In the expanding phase II, the d-modes of Φ, φ_χ, φ_δ, φ_v and φ_κ in eqs. (43-45) decay away whereas C-modes remain constant and have the roles of the relatively growing modes. Whereas, as $t \to t_I$ in the contracting phase although the C-modes remain constant, the d-modes diverge (for $-1 < w_I < 1$) as

$$\Phi \propto \varphi_v \propto \varphi_\delta \propto \varphi_\kappa \propto \frac{1}{a^3 H} \propto |t - t_I|^{-\frac{1-w_I}{1+w_I}},$$
$$\varphi_\chi \propto \frac{H}{a} \propto |t - t_I|^{-\frac{5+3w_I}{3+3w_I}}. \tag{49}$$

For $w_I = 0$, the d-mode of Φ vanishes exactly, and the d-mode of φ_v vanishes in near flat situation. For $w_I = 1$, the d-modes of Φ, φ_v, φ_δ and φ_κ show $\ln|t - t_I|$ divergence, instead. A complete set of solutions in several different gauge conditions is presented in Tables 2-5 of [25]; although the solutions were derived in the expanding phase, the same solutions remain valid in the collapsing phase with the time replaced by its absolute value.

A simple example of the bounce is the case with $K > 0$ and a positive Λ [34]

$$a_B(t) = \sqrt{3K/\Lambda} \cosh(\sqrt{\Lambda/3}\,t). \tag{50}$$

Evolution of the scale factor is plotted in Fig. 2. Either for the vanishing fluid with pure Λ, or for a Λ-type fluid, we have $\mu + p = 0$. If we have $\mu + p = 0$ strictly the basic set of perturbation equations becomes trivial and we cannot determine the perturbations properly, i.e., we do not have meaningful perturbations.
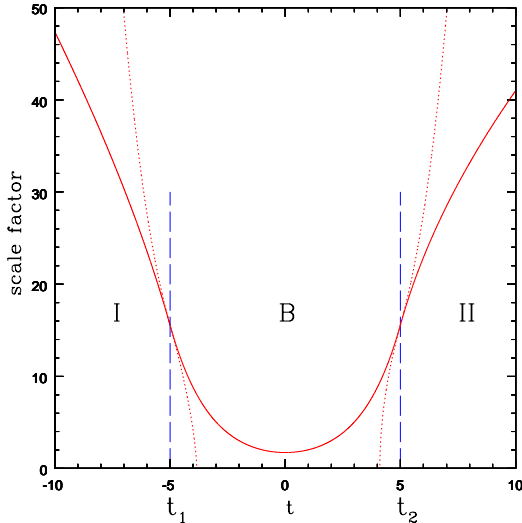
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FIG. 2. Evolution of the scale factor (line). The dotted lines indicate the extensions of the collapsing, bouncing and expanding phases without matchings. The bouncing phase lasts for $|t_B| < 5$ with $\Lambda = 1$ and $K = 1$. As an example, we take $w_I = 0$ and $w_{II} = \frac{1}{3}$.

In the following we consider the perturbation evolutions in three other examples of the bouncing phase. The first two models rely on the fluid/field which give effectively $w < -\frac{1}{3}$ equation of state during the bounce. To have the bounce in such models the positive curvature should have significant role during the bounce, thus not suitable for the bouncing model assumed in §IV. In addition, as the background curvature becomes important, all the perturbation scales go through the small-scale regime where the pressure gradient term becomes important. The third model relies on a presence of an exotic matter which gives a negative contribution to the total energy density. In this case we have a bounce without resorting to the positive curvature, thus the scales remain in the large-scale during the bounce and the model suits the requirement of the bounce assumed in §IV.

A. Bounce with a $w = -\frac{2}{3}$ fluid

For an ideal fluid with $w = $ constant, $K > 0$ and $\Lambda = 0$, from the Friedmann equation we have $H = 0$ at $a(t_*) = (2c_0/K)^{1/(1+3w)}$ where $c_0 \equiv (4\pi G/3)\mu a^{3(1+w)}$. We can show that $a(t_*)$ is a maximum for $w > -\frac{1}{3}$, and is a minimum for $w < -\frac{1}{3}$. As a simple example which gives a bounce we consider $w = -\frac{2}{3}$ case, [35]. Although it is uncertain whether it is appropriate to consider an ideal fluid for $w < 0$ case, we will take the ideal fluid assumption, see a cautionary remark in §VII of [9]. We will find a fundamentally different result in a more realistic (in the sense that we have the concrete action and equation of motion) case based on a scalar field, see §V B.

Equation (5) gives an exact solution

$$a = (c_0/2)(t^2 + K/c_0^2). \tag{51}$$

We have $c_s^2 = -\frac{2}{3}$ and

$$\frac{\ddot{x}}{x} = \frac{c_0^2}{2a^2}\frac{3t^4 + (K/c_0^2)^2}{t^2}, \quad \frac{\ddot{y}}{y} = \frac{c_0^2}{2a^2}(t^2 - 3K/c_0^2). \tag{52}$$

The pressure terms become important compared with the gravity near the bounce. Thus, in order to follow the evolutions we need to handle perturbations based on the full equations. Equation (14) gives

$$\ddot{\varphi}_\chi + \frac{4t}{t^2 + K/c_0^2}\dot{\varphi}_\chi - \frac{8}{3c_0^2}\frac{k^2 - 3K}{(t^2 + K/c_0^2)^2}\varphi_\chi = 0. \tag{53}$$

Ignoring the k^2 term the exact solutions in eqs. (16,17) can be integrated, and for φ_χ we have

$$\varphi_\chi = C\frac{1}{3}\frac{t^4 + 6(K/c_0^2)t^2 - 3(K/c_0^2)^2}{(t^2 + K/c_0^2)^2}$$
$$+ d\frac{4t/c_0}{(t^2 + K/c_0^2)^2}. \tag{54}$$

Since the k^2 terms become negligible away from the bounce we can use these solutions as the proper initial conditions for the C- and d-modes. A typical evolution is presented in Fig. 3.
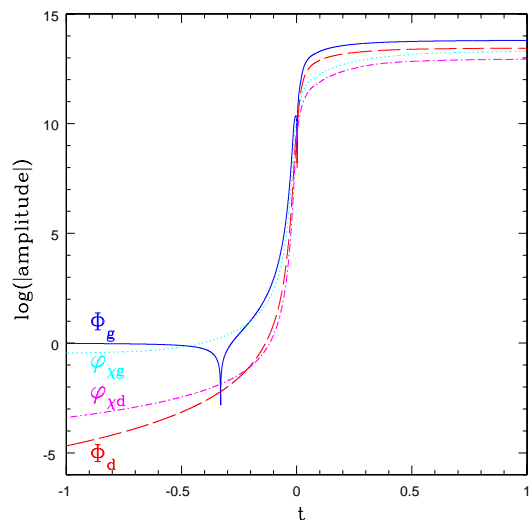

FIG. 3. Evolutions of Φ for the C-mode (blue, line) and d-mode (red, long-dashed line) initial conditions, and φ_χ for the C-mode (cyan, dotted line) and d-mode (magenta, dot-short-dashed line) initial conditions. We take $n = 10$. The pressure terms become important in $|t| < .4$ for φ_χ and in $|t| < .2$ for Φ. Notice that the C-mode of Φ changes sign twice, whereas the d-mode changes once. The sign changes in the expanding phase occur at the same time.

As we have $c_s^2 = -\frac{2}{3}$ we anticipate an exponential growth/decay of the perturbation while the pressure gradient term becomes important. In Fig. 3 both the C- and d-modes in the contracting phase become the (relatively growing) C-mode in the expanding phase. As the large-scale conditions are violated both the C- and d-modes will be dominated by the exponentially growing mode in the small-scale limit, and eventually we cannot trace the C- and d-modes in the expanding phase to the ones in the contracting phase.

B. Bounce with a massive scalar field

We consider a bouncing model based on a massive minimally coupled scalar field with a positive curvature [36]. Results up to eq. (18) in §II remain valid for the field with a prescription mentioned below eq. (9). The background equations are presented in eq. (5) and below it with $\mu_\phi = \frac{1}{2}(\dot{\phi}^2 + m^2\phi^2)$, etc. Equation (14) gives

$$\ddot{\varphi}_\chi + \left(7H + 2\frac{m^2\phi}{\dot{\phi}}\right)\dot{\varphi}_\chi$$
$$+ \left(\frac{m^2\phi^2}{M_{pl}^2} + 2H\frac{m^2\phi}{\dot{\phi}} + \frac{k^2 - 8K}{a^2}\right)\varphi_\chi = 0, \qquad (55)$$

where $M_{pl}^2 \equiv 1/(8\pi G)$. Once we have φ_χ, the rest of perturbations Φ, φ_v, φ_δ and φ_κ follow from eqs. (6,7,10,11). We have

$$c_A^2\frac{k^2}{a^2} = \frac{2m^2\phi}{H\dot{\phi}}\frac{K}{a^2} + \frac{k^2}{a^2}, \qquad (56)$$

$$\frac{\ddot{y}}{y} = \frac{2m^2\phi}{H\dot{\phi}}\frac{K}{a^2} + m^2 - \frac{3}{4}\frac{K}{a^2} + \frac{7\dot{\phi}^2 + 25m^2\phi^2}{24M_{pl}^2}$$
$$+ 2m^2\frac{\phi}{\dot{\phi}}\left(m^2\frac{\phi}{\dot{\phi}} + 4H\right). \qquad (57)$$

We can show that near the bouncing era the pressure term in eq. (56) dominates over the gravity term in eq. (57); this is true even for $k^2 = 0$. Thus, near the bounce the large-scale condition is violated, and with the positive sign in front of k^2 term in eq. (56) we can show that perturbations show oscillatory behavior while in the small scale, see Fig. 4. Although the first term in the RHS of eq. (56) diverges near the bounce, the same term appears in the gravity part in eq. (57) as well. Due to the positive sign in the second term we expect oscillatory instability as the term dominates the gravity part. Near the bounce we have $H \simeq 0$, thus $\mu_\phi \simeq$ constant, and

$$a \simeq \sqrt{3M_{pl}^2 K/\mu_\phi} \cosh\left[\sqrt{\mu_\phi/(3M_{pl}^2)}t\right]. \qquad (58)$$
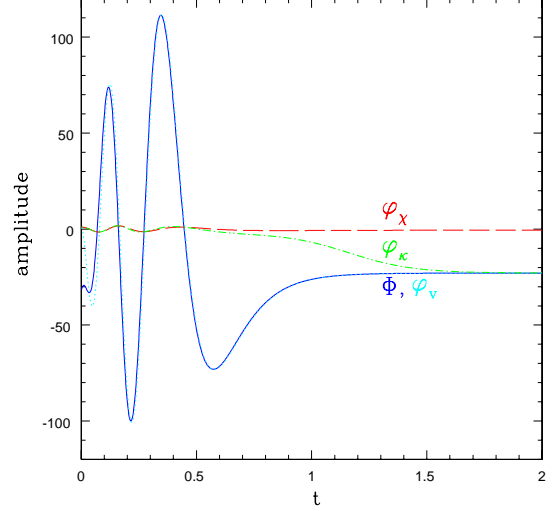

FIG. 4. Evolutions of φ_χ (red, long-dashed line), Φ (blue, line), φ_v (cyan, dotted line), and φ_κ (green, dot-short-dashed line). We omit φ_δ which has a larger amplitude and approaches Φ in the large-scale. We take $n = 10$. $t = 0$ corresponds to the minimum of the bounce, and the pressure dominates over the gravity till $t \sim 1$. We can show that Φ, φ_v, φ_κ and φ_δ stay constant in the large-scale, whereas φ_χ still adjusting its value even in the large-scale as the background equation of state effectively changes from $w_\phi \simeq -1$ while ϕ is rolling to $w_\phi = 0$ as ϕ starts oscillating which occurs for $t > 11$.

In this model, during the bounce all scales reach the small-scale where we cannot apply our large-scale solutions. In Fig. 4 we used an arbitrary initial condition at the minimum of the bounce ($t = 0$), and as the scale becomes large-scale we have only the C-mode because the d-mode in expanding phase is decaying (thus transient) and yields to the relatively growing C-mode within a few expansion time scale. Thus, as in the previous example based on the $w = -\frac{2}{3}$ fluid the C- and d-modes during the collapsing phase are mixed up while in the small-scale, and we cannot trace the C- and d-modes in the expanding phase to the ones in the collapsing phase. One important difference of the scalar field compared with the $w = -\frac{2}{3}$ fluid is that, while fluctuation of the fluid shows exponential instability due to the negative c_s^2 term, the field fluctuation shows oscillatory behavior [37]. Such a difference comes from the presence of a nonvanishing entropic perturbation term e in the case of a scalar field, see below eq. (9).

In later expanding phase as ϕ starts oscillating near potential minimum the background model enters an era with effectively $w_\phi = 0$ (dust) equation of state. As ϕ starts oscillating we cannot solve eq. (55) directly. Instead, we can analytically handle the situation using proper time averaging over the coherent oscillations of the background and the perturbed field [38]. In [39] it

was shown that while the background enters the dust era, the perturbations also behave like a cold dark matter even in the large-scale limit.

C. Bounce model with $\mu = \mu_m - \mu_X$

The positively curved FLRW world model with only the radiation and matter does not allow bouncing after the big crunch. If the physical state near the big crunch allows a presence of additional matter X with its effective energy density behaving as $-\mu_X(t) = -\mu_{X0}a^{-3(1+w_X)}$ and $w_X > \frac{1}{3}$, we could have a smooth and non-singular bouncing phase; this is a generalized case of a 'desparate' example mentioned in p368 of [3]. Thus, for the bounce only, we do not even need the positive curvature in the background world model.

As a toy model allowing such a smooth and non-singular transition with the relevant scale satisfying the large-scale condition we consider a case with the pressure-less matter and the exotic matter with $w_X = \frac{1}{3}$. Thus, we consider a model with the dust and the radition with a negative sign in the radiation component.

$$H^2 = \frac{8\pi G}{3}(\mu_m - \mu_X) - \frac{K}{a^2}, \qquad (59)$$

where $\mu_m \propto a^{-3}$ and $\mu_X \propto a^{-4}$. Certainly, this is not a realistic model for the bounce because we need to assume that there is no conventional radiation component present; to be more realistic, the sum of the radiation and the X-matter should give a net negative contribution. Later we will show, however, that this toy model captures the basic physics of more realistic situations.

For a positive curvature, $K > 0$, eq. (59) gives an exact solution

$$a = \frac{1}{K}c_m \left[1 - \sqrt{1 - 2(c_X/c_m^2)K} \cos\left(\sqrt{K}\eta\right)\right], \qquad (60)$$

where $c_X \equiv (4\pi G/3)\mu_X a^4$; c_X/c_m^2 is dimensionless. We have normalized the time axis so that a has minimum values at $\eta = 2\bar{n}\pi/\sqrt{K}$ with $\bar{n}$ an integer number. For vanishing X-component, $c_X = 0$, we recover the solution in eq. (32). With the X-component the model shows a cyclic behavior. The basic picture of the cyclic bounces remains valid in more realistic situations with $w_X > \frac{1}{3}$.

The K term becomes important near $a_{\max}$, and near $a_{\min}$ we have $\mu_X \simeq \mu_m$. The curvature term is negligible near the bounce, thus allowing existence of the large-scale where we could ignore the Laplacian term coming from the pressure gradient. If we do not need a cyclic behavior we can actually take a flat model with $K = 0$. Assuming negligible K term we have

$$a \simeq \frac{c_m}{2}\left(\eta^2 + 2c_X/c_m^2\right), \qquad (61)$$

which is an exact solution of eq. (59) for $K = 0$.

It is not entirely clear how to handle the perturbation of the exotic component X which is introduced to have a bounce in the background without resorting to the positive curvature. If such an exotic state of matter can be modelled by using a field or modifying terms in the gravity theory (coming from quantum corrections or higher dimensions, for example), we need the correct forms of the equation of motion or the modified action to handle the perturbations properly. In the present situation lacking the concrete model for the exotic matter, we will take a phenomenological approach based on a fluid approximation.

Let us *assume*, except for its negative contribution to the density, the fluid X behaves as an ordinary ideal fluid with an equation of state w_X. In such a case, near the bounce we need to consider two-component system with m and X. [The other possibility is to derive solutions in the m- and X-dominated eras separately and to connect by using the matching conditions. As the X-fluid cannot dominate the total fluid (we have $\mu_m \geq \mu_X$ with the equality holding at the bounce) such a situation is forbidden.]

We use the conventional decomposition of the system into the adiabatic (curvature) perturbation, characterized by φ_χ (or Φ), and the relative (often called the isocurvature) perturbation defined as $S \equiv S_{12} \equiv \delta_1/(1+w_1) - \delta_2/(1+w_2)$, [40]. The basic equations in the two-component system become, see eqs. (23,35,46,57) in [13]

$$\frac{\mu + p}{H}\left[\frac{H^2}{a(\mu + p)}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\varphi_\chi = -4\pi Ge$$

$$= -4\pi G\frac{(\mu_1 + p_1)(\mu_2 + p_2)}{\mu + p}(c_{(1)}^2 - c_{(2)}^2)S, \qquad (62)$$

$$\ddot{S} + H\left(2 - 3c_z^2\right)\dot{S} + c_z^2\frac{k^2}{a^2}S$$

$$= -\frac{k^2(k^2 - 3K)}{a^4}\frac{c_{(1)}^2 - c_{(2)}^2}{4\pi G(\mu + p)}\varphi_\chi, \qquad (63)$$

where $c_{(i)}^2 \equiv \dot{p}_i/\dot{\mu}_i$, and

$$c_z^2 \equiv \frac{c_{(2)}^2(\mu_1 + p_1) + c_{(1)}^2(\mu_2 + p_2)}{\mu + p}. \qquad (64)$$

In our case $1 = m$ and $2 = X$, thus, $w_1 = c_{(1)}^2 = 0$, $w_2 = c_{(2)}^2 = \frac{1}{3}$, $\mu + p = \mu_m - \frac{4}{3}\mu_X$, etc. In the large-scale limit the curvature mode can be sourced by the isocurvature one, see eq. (62), whereas the isocurvature mode decouples from the curvature mode in general [41], see eq. (63). [The situation is different in the case of multiple number of scalar field: in such a case, effectively, the RHS of eq. (63) has k^2/a^2 factor instead of k^4/a^4, thus the isocurvature modes are less decoupled from the adiabatic one, see [13,42].]

Let us *assume* an adiabatic initial condition, thus setting $S = 0$ at early era in the contracting phase, for

simplicity. More precisely, we are assuming $S \ll \varphi_\chi$ in the initial epoch which is natural because it means we are assuming no significant fluctuations in the X-component in the early matter dominated era. Since the curvature mode does not source the isocurvature mode in the large-scale, the isocurvature mode will remain small. In such a case the RHS of eq. (62) vanishes, and the curvature equations in the single component situation, eq. (14), remain valid without any change. Thus, for scales satisfying the large-scale limit we have the same solutions in eqs. (16,17) remaining valid. The solutions become:

$$\Phi = C + \tilde{d}k^2 \frac{64}{27} \frac{c_X}{c_m^2} \int^\eta \left(\eta^4 + \frac{4c_X}{3c_m^2}\eta^2 - \frac{4c_X^2}{3c_m^4} \right)^{-2} \eta^2 d\eta,$$

$$\varphi_\chi = \left[\left(1 + \frac{c_m^2}{2c_X}\eta^2 + \frac{5c_m^4}{12c_X^2}\eta^4 + \frac{3c_m^6}{40c_X^3}\eta^6 \right) C \right.$$

$$\left. + \frac{c_m^6}{c_X^3}\eta\tilde{d} \right] \Big/ \left(1 + \frac{c_m^2}{2c_X}\eta^2 \right)^3 . \quad (65)$$

For φ_χ the contribution from the lower bound of integration of the C-mode is absorbed to the $\tilde{d}$-mode. In the matter-dominated eras, $|\eta| \gg 2c_X/c_m^2$, we have

$$\Phi = C - k^2 \frac{64}{135} \frac{c_X}{c_m^2}\eta^{-5}\tilde{d}, \quad \varphi_\chi = \frac{3}{5}C + 8\eta^{-5}\tilde{d}. \quad (66)$$

Near the big bang/crunch, the solution for φ_χ coincides with the one for a pressureless medium considered in eq. (38). For $c_X/c_m^2 \to 0$, Φ also coincides with the one known in the pressureless medium. Near the bounce, $|\eta| \ll 2c_X/c_m^2$, we have

$$\Phi = C + k^2 \frac{4}{9}\frac{c_m^6}{c_X^3}\eta^3\tilde{d}, \quad \varphi_\chi = C + \frac{c_m^6}{c_X^3}\eta\tilde{d}, \quad (67)$$

which are regular and finite.

Since the present bounce model allows the scales to stay in the large-scale limit during the transition, it can be considered as a concrete model of the smooth and non-singular bounce assumed in §IV. Indeed, the curvature term is negligible near the bounce as it was the case near the big crunch/bang in §IV. In the $c_X/c_m^2 \to 0$ limit, eq. (65) reduces to eq. (66) which also coincides with the known solution considered in §IV.

Apparently, we can also make a more realistic model with the radiation, matter and X where $w_X > \frac{1}{3}$. We note that eqs. (62,63) remain valid for any two-component system of matter perturbations. Even in such a case the X fluid can cause a smooth and non-singular bounce and the curvature term has negligible role near the bounce. Thus, essentially the same conclusions (e.g., the C-mode feeding the C-mode) remain valid. We have considered a simple toy model only because it allows analytic handling of the background and the perturbations, thus showing the situation explicitly.

VI. DISCUSSIONS

In §IV we have shown that the perturbation in a positively curved FLRW model filled with a pressureless matter is described by the conservation of Φ. Assuming a transition of big crunch followed by a big bang in such a model, by using the known matching conditions we have shown that Φ maintains the same value even after the transition. Using the matching conditions we also have shown that the diverging solution in the contracting phase is matched to the decaying solution in the subsequent expanding phase, whereas the other solution which stays constant during the contracting phase is matched into the same constant solution in the expanding phase. That constant mode is characterized by Φ; the other solution of Φ which decays in expanding phase is higher-order in k^2 compared with the one of φ_χ and vanishes for the vanishing background pressure.

In order to confirm these results based on the matching at singular bounce, in §V we have considered three different non-singular and smooth bounce models. For the bounce models based on the fluid (§V A) and the massive scalar field (§V B) in the positively curved background, the role of background curvature is important to make the bounce. In such cases, all the perturbation scales come inside the sound-horizon near bounce, and the large-scale conditions are violated. As the pressure gradient terms become important, perturbations in the fluid model show exponential instability, whereas the ones in the massive field model show oscillatory behaviors. For both situations the two independent perturbation modes in the large-scale limit during the collapsing phase got mixed up with the two independent modes in the small-scale during the bouncing phase. Thus, we cannot trace the two independent solutions (C- and d-modes) in the expanding phase to the ones in the contracting phase.

In §V C we considered a bouncing model based on an exotic matter with negative contribution to the total energy density. In such a case the positive curvature is not important during the bounce. Thus, we could have the relevant scales remaining in the large-scale limit, and could apply the general large-scale solutions. In this case, however, we have to handle the perturbation of the exotic matter in addition to the ordinary one simultaneously. By considering the adiabatic initial condition we have shown that the same curvature perturbation equation known in the single-component situation remains valid, thus the known large-scale solutions are valid as well throughout the bounce. Therefore, this third-type of bounce model can be regarded as an example of the smooth and non-singular bouncing assumed in §IV. As an analytically manageable concrete example, we have considered a simple toy model with a dust and an exotic matter with the radiation-like equation of state. Even in more general situations of the equation of states before and after the bounce, similar analyses can be made which

show that the C-mode in the expanding phase is affected only by the C-mode in the contracting phase, thus the growing mode in the contracting phase decays away as the world model enters expanding phase.

Our analyses are based on two assumptions: (i) the contracting phase is converted into the expanding one by a smooth and non-singular bounce, and (ii) the linear perturbation theory holds during the evolution. The large-scale evolution can be characterized by the conservation of Φ. We have shown that the C-mode of Φ, which is the proper growing mode in expanding phase, is simply conserved during the evolution and through bounces. The results are true as long as the two assumptions made above are valid, and in addition, if the large-scale condition is met during the transition as considered in §V C.

In §III we showed that the three dimensionless measures, the intrinsic curvatures (φ and $C^{(t)}_{\alpha\beta}$), the tracefree part of the extrinsic curvature ($\hat{\sigma}/H$), and the Weyl curvature (E/R), *diverge* at singularity for $-1 < w < 1$. Thus, for $-1 < w < 1$, the spacetime perturbations become singular as the background approaches the singularity. An ambiguity remains for $w = 1$ case because although φ and $C^{(t)}_{\alpha\beta}$ diverge logarithmically, $\hat{\sigma}/H$ and E/R remain finite. These apply to all perturbation types and for all gauge conditions we have considered. Behaviors of the other variables (the perturbed lapse function α, a dimensionless measure of the perturbed expansion κ/H, relative density perturbation δ, etc.) depend more strongly on the gauge conditions, see Tables 2 and 4 of [25]. Thus, these variables apparently have less physical significance to characterize the spacetime fluctuations compared with the other three measures whose behaviors are gauge independent at least in the pool of gauge conditions we have investigated. Do above results imply diverging spacetime fluctuations for $-1 < w < 1$, and regular ones for $w = 1$? In Table 4 of [25] we find that in no gauge condition we have *all* the perturbations remain finite for $-1 < w \leq 1$.

The authors of [43] argued that as the model goes through a singular bounce the perturbation becomes nonlinear. We have shown that *if* the fluctuations survive the bounce as linear ones, the diverging mode in the contracting phase should be matched to the decaying one in the expanding phase. Lyth in [44] made the following simple and powerful argument. As we have under the gauge transformation ($\tilde{x}^a = x^a + \xi^a$ with $\xi^t \equiv a\xi^\eta$)

$$\tilde{\varphi} = \varphi - H\xi^t, \quad \tilde{\delta} = \delta + 3H(1+w)\xi^t, \qquad (68)$$

if φ diverges while δ remains finite, or vice versa, in any single gauge condition [this is the case for $-1 < w \leq 1$, see eqs. (9,49) for φ_v and δ_v] no temporal gauge-transformation ξ^t can be found which makes both φ and δ finite. Therefore, for $-1 < w \leq 1$ we encounter the d-mode perturbations of Friedmann world model becoming *singular* near big-crunch in one form or another in *all* gauge conditions.

We note that Φ, which becomes φ_v for $K = 0$, simply stays constant in a pressureless medium, thus its magnitude *cannot* characterize the breakdown of linearity of the perturbation. As we have from eqs. (6,37,38) $\varphi_v = C + (1/3)(1 - \cos\eta)\varphi_\chi$ where we set $K = 1$, φ_v itself could diverge near singularity. From eq. (11), near big crunch in the pressureless medium we have $\varphi_\kappa \simeq \varphi_\delta$ where φ_δ, given in eq. (39), has diverging part. Thus, near big crunch the diverging modes behave as

$$\varphi_v \propto \varphi_\delta \propto \varphi_\kappa \propto |\eta|^{-3}, \quad \varphi_\chi \propto |\eta|^{-5}, \qquad (69)$$

whereas Φ has no diverging mode in the pressureless case. For the situation with general w, see eq. (49). Bardeen has argued that the behavior of φ_χ "overstates the physical strength of the singularity", see below eq. (5.12) in [9].

At the singular big crunch, we certainly have d-modes of many perturbation variables unambiguously becoming singular for $-1 < w \leq 1$, see Tables 2-4 in [25]. Do large amplitudes of perturbations imply breakdown of linear theory? Due to the gauge dependence of relativistic perturbation, large (larger than unity, say) amplitudes of some dimensionless measures of gauge-invariant perturbation variables do not guarantee the breakdown of the linear theory. However, what Lyth [44] has shown is that in the collapsing phase we could encounter situations where the amplitudes of perturbation variables become large in one form or the other in *all* gauge conditions. Lyth has argued this as the violation of the *necessary condition* of the linear perturbation theory.

In our models avoiding the singularity by a smooth and nonsingular bounce it is likely that certain scales can safely go through the bounce retaining their linear nature. As we have assumed the linearity of perturbations, our analyses and results are applicable to such scales only. In §IV and V C we have shown that the diverging solutions in the contracting phase in eq. (69) affect only the decaying, thus transient, solutions in the subsequent expanding phase. In such a scenario, however, one could anticipate large (compared with the C-mode) amount of the decaying (d) mode present in the early big-bang phase for a while which is the remnant from the preceding phase before the big-bang.

In a recently proposed ekpyrotic scenario based on colliding branes [45] it was argued that the final scalar-type perturbation is scale invariant [46]. In [47] it was shown that the generated scale-invariant spectrum in the zero-shear gauge during the collapsing phase should be identified as the d-mode, thus after the bounce we have a different power-spectrum [48]. Our results in this paper confirm that while the large-scale condition is met during the (smooth and non-singular) transition the d-mode in the contracting phase does not affect the (properly growing) C-mode in the expanding phase. The background curvature is flat in the ekpyrotic scenario and the scale remains in the large-scale during the bounce. However, since the bounce of the ekpyrotic scenario goes through

a singularity the author of [44] has argued that one cannot rely on the linear analyses as the model approaches the singularity. Thus, either the final spectrum is not scale-invariant (which is the case if the linear perturbation survives) or the issue should be handled in the future string theory context with a concrete mechanism for the bounce.

Acknowledgments

[1] R.C. Tolman, Phys. Rev. **38**, 1758 (1931); G. Lemaître, Ann. Soc. Sci. Bruxelles A **53**, 51 (1933), translated in Gen Rel. Grav. **29**, 935 (1997); M.J. Rees, The Observatory **89**, 193 (1969); R. Dicke and P.J.E. Peebles, in: *General relativity*, eds., S.W. Hawking and W. Israel, (Cambridge Univ. Press, Cambridge, 1979), 504p; Ya.B. Zel'dovich and I.D. Novikov, *Relativistic astrophysics, Vol 2, The structure and evolution of the universe* (Univ. Chicago Press, Chicago, 1983); G.F.R. Ellis, Ann. Rev. Astron. Astrophys. **22**, 157 (1984); J.D. Barrow and F.J. Tipler, *The Anthropic cosmological principle* (Clarendon Press, Oxford, 1986); J.D. Barrow and M.P. Dąbrowski, Mon. Not. R. Astron. Soc. **275**, 850 (1995).

[2] R.C. Tolman, *Relativity, thermodynamics and cosmology*, (Oxford Univ. Press, London, 1934).

[3] P.J.E. Peebles, *Principles of physical cosmology* (Princeton Univ. Press, Princeton, 1993).

[4] H. Nariai, Prog. Theor. Phys. **46**, 433 (1971); H. Nariai and K. Tomita, *ib id.* **46**, 776 (1971); L. Parker and S.A. Fulling, Phys. Rev. D **7**, 2357 (1973); G. L. Murphy, *ib id.* **8**, 4231 (1973); H. Nariai, Prog. Theor. Phys. **51**, 613 (1974); J.D. Bekenstein, Phys. Rev. D **11**, 2072 (1975); V.N. Melnikov and S.V. Orlov, Phys. Lett. A **70**, 263 (1979); M. Visser, Phys. Lett. B **349**, 443 (1995), (gr-qc/9409043).

[5] C. Molina-París and M. Visser, Phys. Lett. B **455**, 90 (1999), (gr-qc/9810023).

[6] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt, and N. Turok, hep-th/0108187; P.J. Steinhardt and N. Turok, hep-th/0111098; G. Felder, A. Frolov, L. Kofman, and A. Linde, hep-th/0202017.

[7] J.M. Bardeen, in: *Cosmology and particle physics*, eds., L. Fang and A. Zee, (Gordon and Breach, London, 1988), 1; J. Hwang, Astrophys. J. **375**, 443 (1991).

[8] J. Hwang and H. Noh, Phys. Rev. D **65** 023512 (2002), (astro-ph/0102005).

[9] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[10] G.B. Field and L.C. Shepley, Astrophys. Space. Sci. **1**, 309 (1968); G.B. Field, in *Stars and stellar system, Vol IX, Galaxies and the universe*, eds., A. Sandage, M. Sandage, and J. Kristian, (Univ. of Chicago Press, 1975) Chicago), 359; G.V. Chibisov and V.F. Mukhanov, Mon. Not. R. Astron. Soc. **200**, 535 (1982).

[11] J. Hwang and E.T. Vishniac, Astrophys. J. **353**, 1 (1990).

[12] J. Hwang, Phys. Rev. D **60**, 103512 (1999), (astro-ph/9907080).

[13] J. Hwang and H. Noh, Class. Quant. Grav. in press (2002), astro-ph/0103244.

[14] A.R. Liddle and D.H. Lyth, *Cosmological inflation and large-scale structure*, (Cambridge Univ. Press, Cambridge, 2000).

[15] V.N. Lukash, Sov. Phys. JETP Lett. **31**, 596 (1980); Sov. Phys. JETP **52**, 807 (1980).

[16] V.F. Mukhanov, JETP Lett. **41**, 493 (1985); V.F. Mukhanov, Soviet Phys. JETP **68**, 1297 (1988); V.F. Mukhanov, H.A. Feldman, and R.H. Brandenberger, Phys. Rep. **215**, 203 (1992).

[17] E.M. Lifshitz, J. Phys. (USSR) **10**, 116 (1946); E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. **12**, 185 (1963); L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon, Oxford, 1975) §115.

[18] D.H. Lyth and A. Woszczyna, Phys. Rev. D **52**, 3338 (1995), (astro-ph/9501044).

[19] J. Hwang and E.T. Vishniac, Astrophys. J. **382**, 363 (1991).

[20] N. Deruelle and V.F. Mukhanov, Phys. Rev. D **52**, 5549 (1995), (gr-qc/9503050); J. Martin and D.J. Schwarz, Phys. Rev. D **57**, 3302 (1998), (gr-qc/9704049).

[21] E. Harrison, Astrophys. J. **383**, 60 (1991).

[22] A.P. Lightman, and W.H. Press, Astrophys. J. **337**, 598 (1989).

[23] E.R. Harrison, Phys. Rev. D **1**, 2726 (1970); D.H. Lyth, *ib id.* **31**, 1792 (1985); D.H. Lyth, and M. Mukherjee, *ib id.* **38**, 485 (1988); J. Hwang, Astrophys. J. **380**, 307 (1991); J. Hwang, and J. J. Hyun, *ib id.* **420**, 512 (1994).

[24] A.A. Starobinsky, S. Tsujikawa, and J. Yokoyama, Nucl. Phys. B **610** 383 (2001), (astro-ph/0107555).

[25] J. Hwang, Astrophys. J. **415**, 486 (1993).

[26] J. Ehlers, Gen. Rel. Grav. **25**, 1225 (1993), translated from German originally published in Akad. Wiss. Lit. Mainz, Abhandl. Math.-Nat. Kl. **11** 792 (1961); G.F.R. Ellis, in *General relativity and cosmology, Proceedings of the international summer school of physics Enrico Fermi course 47*, edited by R. K. Sachs (Academic Press, New York, 1971), 104; G.F.R. Ellis, in *Cargese Lectures in Physics*, edited by E. Schatzmann (Gorden and Breach, New York, 1973), 1.

[27] R. Durrer, Fund. Cosmic Phys. **15**, 209 (1994), (astro-ph/9311041).

[28] R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov, and G. Veneziano, Phys. Rev. D **51**, 6744 (1995), (hep-th/9501066).

[29] A. Friedmann, Zeitschrift für Physik **10**, 377 (1922), translated in: *Cosmological constants, papers in modern cosmology*, eds., J. Bernstein and F. Feinberg, (Columbia Univ. Press, New York, 1986), 49p.

[30] J. Jeans, Phil. Trans. Roy. Soc. **199A**, 49 (1902); W.B. Bonnor, Mon. Not. R. Astron. Soc. **107**, 104 (1957); E.R. Harrison, Rev. Mod. Phys. **39**, 862 (1967).

[31] H. Nariai, Prog. Theor. Phys. **41**, 686 (1969); K. Sakai, *ib id*. **41**, 1461 (1969).

[32] §15.9 in S. Weinberg, *Gravitation and cosmology* (Wiley, New York, 1972); §11 in P.J.E. Peebles, *The large-scale structure of the universe* (Princeton Univ. Press, Princeton, 1980).

[33] E.J. Groth and P.J.E. Peebles, Astron. Astrophys. **41**, 143 (1975).

[34] W. de Sitter, Mon. Not. R. Astron. Soc. **LXXVIII**, 3 (1917).

[35] Ya.B. Zel'dovich, L.B. Okun, and I.Yu. Kabzarev, Sov. Phys. JETP **40**, 1 (1975).

[36] J.D. Barrow and R.A. Matzner, Phys. Rev. D **21**, 336 (1980).

[37] S. Gratton, A. Lewis, and N. Turok, astro-ph/0111012.

[38] B. Ratra, Phys. Rev. D **44**, 352 (1991).

[39] J. Hwang, Phys. Lett. B **401**, 241 (1997), (astro-ph/9610042).

[40] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

[41] H. Kodama and M. Sasaki, Int. J. Mod. Phys. A **1**, 265 (1986).

[42] J. Hwang and H. Noh, Phys. Letts. B **495**, 277 (2000), (astro-ph/0009268).

[43] P. Peter and N. Pinto-Neto, Phys. Rev. D **65**, 023513 (2002), (gr-qc/0109038).

[44] D.H. Lyth, hep-ph/0110007.

[45] C. Park and S-J Sin, Phys. Lett. B **485**, 239 (2000), (hep-th/0005013); J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, Phys. Rev. D **64** 123522 (2001), (hep-th/0103239).

[46] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, hep-th/0109050; R. Durrer, hep-th/0112026.

[47] R. Brandenberger and F. Finelli, JHEP **0111**, 056 (2001), (hep-th/0109004); J. Hwang, Phys. Rev. D in press (2002), (astro-ph/0109045); S. Tsujikawa, gr-qc/0110124; F. Finelli and R. Brandenberger, hep-th/0112249; J. Martin, P. Peter, N. Pinto-Neto, D.J. Schwarz, hep-th/0112128.

[48] D.H. Lyth, Phys. Lett. B **524**, 1 (2002), (hep-ph/0106153).